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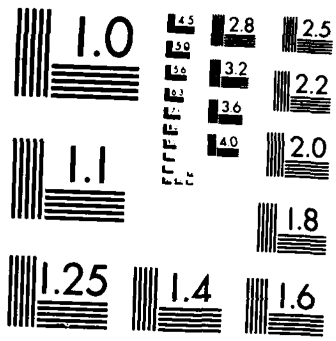

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CONTINUUM STRUCTURE FUNCTIONS

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SUMMARY

A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. The theory of such functions generalizes the traditional theory of binary and multistate structure functions, permitting more realistic and flexible modelling of systems subject to reliability growth, component degradation and partial availability.

During the two years of work on this topic, the PI has developed a theory of modules (i.e. subsystems), calculated various sets of bounds on the distribution of the structure function when the component states are random variables, ~~deduced~~ axiomatic characterizations of two important special cases, ~~derived~~ a definition of the reliability importance of the various components, and ~~deduced~~ a theory of cannibalization.

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1. INTRODUCTION

A continuum structure function (CSF) on the unit hypercube is a mapping γ: $[0,1]^n \rightarrow [0,1]$ which is nondecreasing in each argument; we assume, without any loss of generality, that $\gamma(\underline{0}) = 0$ and $\gamma(\underline{1}) = 1$, writing $\underline{\alpha} = (\alpha,\ldots,\alpha) \in \Delta = [0,1]^n$. Such functions are used to relate the states $\underline{x}$ of the components C of a machine to that of the machine itself, generalizing the well-known theory of binary and multistate structure functions. In this report, we review the PI's research on such functions during the two years of funding under Grant AFOSR-84-0243.

2. MODULES AND BOUNDS

Firstly, we present results which generalize the traditional theory of modules to CSFs and which generalize various bounds on the reliability function to the distribution of $\gamma(\underline{X})$ when $\underline{X}$ is a vector of associated random variables.

Definition

A CSF γ is weakly coherent if $\sup_{\underline{x}}[\gamma(1_i,\underline{x}) - \gamma(0_i,\underline{x})] > 0$ for $i = 1,2,\ldots,n$, writing $(\delta_i,\underline{x}) = (x_1,\ldots,x_{i-1},\delta,x_{i+1},\ldots,x_n)$.

Definition

Suppose that γ is a weakly coherent CSF and that $A \subset C$ is nonempty. Suppose, further, that there exists a weakly coherent CSF $\gamma_1 : [0,1]^{|A|} \rightarrow [0,1]$ and a CSF χ: $[0,1]^{n-|A|+1} \rightarrow [0,1]$ such that $\gamma(\underline{x}) = \chi[\gamma_1(\underline{x}^A),\underline{x}^{A^c}]$ for all $\underline{x}$. Then (A,γ_1) is a module of (C,γ) and A is a modular set of (C,γ).

In the above $\underline{x}^A$ denotes $\{x_i \in \underline{x} \mid i \in A\}$.

The principal tool used in the study of modules is the <u>minimal path sets</u> (MPSs) of a CSF. These are subsets of C which mimic the properties of the minimal path sets of a binary structure. If a CSF is <u>upper simple</u>, it has at least one MPS, each component lies in at least one MPS and no MPS is a proper subset of another MPS.

<u>Theorem</u>

Let γ be an upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Suppose that (A,γ_1) is a module of (C,γ) and that $A \cap T_j \neq \emptyset$ for $j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. Suppose, further, that γ_1 is upper simple. Then the minimal path sets of γ_1 are $A \cap T_1 \ldots, A \cap T_k$.

<u>Theorem</u>

Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Suppose that A is a nonempty subset of C such that $T_j \subset A$ for $j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. Then there exists a weakly coherent CSF γ_A: $[0,1]^{|A|} \to [0,1]$ such that (A,γ_A) is a module of (C,γ).

<u>Theorem</u>

Let γ be an upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Suppose that (A,γ_1) is a module of (C,γ) and that $A \cap T_j \neq \emptyset$ for $j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. Suppose, further, that γ_1 is upper simple. Then $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a minimal path set of γ for $j,\ell = 1,2,\ldots,k$.

(*) Suppose that γ is upper simple with minimal path sets $T_1,\ldots,T_r$. Suppose, further, that A is a nonempty subset of C such that $A \cap T_j \neq \emptyset$ for

$j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. If $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a minimal path set of γ for all $j, \ell = 1,2,\ldots,k$, then A is a modular set of (C,γ) and the associated CSF is upper simple.

Three Modules Theorem

Let γ be an upper simple CSF which satisfies (*). Suppose that A_1, A_2 and A_3 are disjoint, nonempty subsets of C such that $A_1 \cup A_2$ and $A_2 \cup A_3$ are modular sets of (C,γ) and that the associated CSFs are upper simple. Then A_1, A_2, A_3, $A_1 \cup A_3$ and $A_1 \cup A_2 \cup A_3$ are all modular sets of (C,γ). Further, those minimal path sets of γ, $T_1,\ldots,T_r$ say, which intersect $A_1 \cup A_2 \cup A_3$ all intersect each of A_1, A_2 and A_3, or else they all intersect exactly one of these sets.

If γ is a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$, then

$$\gamma(\underline{x}) = \max_{1 \le i \le r} \gamma(\underline{x}^{T_i}, \underline{0}^{T_i^c}).$$

This decomposition is used in the proof of the following result.

Theorem

Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Then, if $X_1,\ldots,X_n$ are associated random variables,

$$\max_{1 \le i \le r} P\{\gamma(\underline{X}^{T_i}, \underline{0}^{T_i^c}) \ge x\} \le P\{\gamma(\underline{X}) \ge x\} \le \coprod_{i=1}^{r} P\{\gamma(\underline{X}^{T_i}, \underline{0}^{T_i^c}) \ge x\}$$

for all $x \in R$.

If we can partition $C = A_1 \cup \cdots \cup A_N$ such that each A_i is a modular set of (C,γ), these bounds may be improved as follows.

Theorem

Suppose that γ is a CSF with modular decomposition $\{\chi,(A_1,\gamma_1),\ldots,(A_N,\gamma_N)\}$ and that $\underline{X}$ is a vector of associated random variables. Let

$Y_j = \gamma_j(\underline{X}^{A_j})$ for $j=1,2,\ldots,N$. If γ and χ are both right-continuous and upper simple with minimal path sets $T_1,\ldots,T_r$ and $\mu_1,\ldots,\mu_p$ respectively,

$$\max_{1\le i\le r} P\{(\underline{X}^{T_i},\underline{0}^{T_i^c}) \ge x\} \le \max_{1\le i\le p} P\{\chi(\underline{Y}^{\mu_i},\underline{0}^{\mu_i^c}) \ge x\} \le P\{\gamma(\underline{X}) \ge x\} \le$$

$$\coprod_{i=1}^{p} P\{\chi(\underline{Y}^{\mu_i},\underline{0}^{\mu_i^c}) \ge x\} \le \coprod_{i=1}^{r} P\{\gamma(\underline{X}^{T_i},\underline{0}^{T_i^c}) \ge x\}.$$

One can similarly define <u>minimal</u> <u>cut</u> <u>sets</u> of CSFs and deduce analogous bounds.

Notation: $P_\alpha = \{\underline{x} \mid \gamma(\underline{x}) \ge \alpha$ whereas $\gamma(\underline{y}) < \alpha$ for all $\underline{y} < \underline{x}\}$

where $\underline{y} < \underline{x}$ means that $\underline{y} \le \underline{x}$ but $\underline{y} \ne \underline{x}$.

Theorem (Block and Savits, 1984)

Let γ be a right-continuous CSF and suppose that $\underline{X}$ is a vector of associated random variables. Then

$$\sup_{\underline{y}\in P_\alpha} P\{\underline{X} \ge \underline{y}\} \le P\{\gamma(\underline{X}) \ge \alpha\} \le \coprod_{\underline{y}\in P_\alpha} P\{\underline{X} \ge \underline{y}\}.$$

If γ admits of a modular decomposition, these bounds may be improved as follows.

Theorem

Let γ be a right-continuous, weakly coherent CSF with modular decomposition $\{\chi,(A_1,\gamma_1),\ldots,(A_N,\gamma_N)\}$ in which each γ_i is continuous and suppose that $\underline{X}$ is a vector of associated random variables. Let

$Z_i = \gamma_i(\underline{X}^{A_i})$, $i=1,2,\ldots,N$. Then

$$\sup_{\underline{y}\in P_\alpha} P\{\underline{X} \ge \underline{y}\} \le \sup_{\underline{w}\in Q_\alpha} P\{\underline{Z} \ge \underline{w}\} \le P\{\gamma(\underline{X}) \ge \alpha\} \le \coprod_{\underline{w}\in Q_\alpha} P\{\underline{Z} \ge \underline{w}\} \le \coprod_{\underline{y}\in P_\alpha} P\{\underline{X} \ge \underline{y}\}.$$

where $Q_\alpha = \{\underline{z} \mid \chi(\underline{z}) \geq \alpha$ whereas $\chi(\underline{y}) < \alpha$ for all $\underline{y} < \underline{z}\}$.

There are analogous bounds using the sets $K_\alpha = \{\underline{x} \mid \gamma(\underline{x}) \leq \alpha$ whereas $\gamma(\underline{y}) > \alpha$ for all $\underline{y} > \underline{x}\}$.

3. AXIOMATIZATION

Let $P_1, \ldots, P_p$ denote the p minimal path sets of a binary coherent structure function. The associated Barlow-Wu CSF is defined as

$$\zeta(\underline{x}) = \max_{1 \leq r \leq p} \min_{i \in P_r} x_i .$$

See Baxter (1984). The following theorem provides an axiomatic characterization of such functions.

Theorem

A CSF γ is of the Barlow-Wu type iff it satisfies the following conditions:

C1 γ is continuous

C2 $P_\alpha \subset \{0,\alpha\}^n$, $0 < \alpha \leq 1$

C3 There is no nonempty open set $A \subset [0,1]^n$ such that γ is constant on A

C4 γ is weakly coherent.

The proof requires a series of propositions which are of some independent interest including the following.

Proposition

If γ satisfies C1, C2 and C3, then $\gamma(\{0,\alpha\}^n) = \{0,\alpha\}$ for all $\alpha \in [0,1]$.

Proposition

If γ satisfies C1, C2 and C3, then $P_\alpha = \alpha P_1$ for all $\alpha \in (0,1]$.

An axiomatic characterization of the Natvig CSF (Baxter, 1986) is also deduced.

4. RELIABILITY IMPORTANCE

Let $h: [0,1]^n \to [0,1]$ be a reliability function. The reliability importance of component i is defined as

$$I(i) = \frac{\partial h(\underline{p})}{\partial p_i} = h(1_i, \underline{p}) - h(0_i, \underline{p})$$

where $p_i = P\{X_i = 1\}$ and where $X_1, \ldots, X_n$ are independent binary random variables. In this section, a definition of the reliability importance of components in a CSF is proposed and some properties are presented.

The motivation for our definition (below) is most readily understood by observing that I(i) is the probability that repairing component i will restore a failed system to the operating state. A possible generalization of I(i) to the continuum case would be to regard part of the unit interval, say $[0,\alpha)$ $(0<\alpha<1)$, as corresponding to the failure states of the system and to regard $[\alpha,1]$ as the operating states, in which case one could define reliability importance to be

$$P\{\gamma(\underline{X}) \geq \alpha | X_i \geq \alpha\} - P\{\gamma(\underline{X}) \geq \alpha | X_i < \alpha\}.$$

Consideration of the CSF $\gamma(x_1, x_2) = x_1 x_2$ suggests that this definition is not wholly satisfactory: if $x_1 = x_2 = \beta \in [\alpha, \sqrt{\alpha})$ $(0<\alpha<1)$, then neither component is in the failed state even though the system itself should be regarded as failed. This difficulty may be circumvented by replacing $\underline{\alpha}$ by a suitably chosen element of ∂U_α where $U_\alpha = \{\underline{x} | \gamma(\underline{x}) \geq \alpha\}$; considerations of symmetry indicate that the vector chosen, called the key vector, should also lie on the diagonal of the unit hypercube.

Definition

Let $H = \{\underline{\alpha} | \underline{0} \leq \underline{\alpha} \leq \underline{1}\}$ be the diagonal of the unit hypercube. We say that the vector $\underline{\delta} = \underline{\delta}(\alpha) = H \cap \partial U_\alpha$ is the key vector of U_α and we call δ the key element.

Theorem

For any CSF γ, the key vector always exists and, if γ is continuous, $\gamma(\underline{\delta}) = \alpha$ for all $\alpha \in (0,1]$.

Since the key vector $\underline{\delta}$ exists for any CSF and for any $\alpha \in (0,1]$, and since Δ is symmetric about H, we define reliability importance as follows.

Definition

The reliability importance $R_i(\alpha)$ of component i at level $\alpha \in \text{Im } \gamma - \{0\}$ for the CSF γ is defined as

$$R_i(\alpha) = P\{\gamma(\underline{X}) \geq \alpha | X_i \geq \delta\} - P\{\gamma(\underline{X}) \geq \alpha | X_i < \delta\}$$

where $\underline{X}$ is a random vector and where δ is the key element of U_α.

Theorem

Suppose that γ is a continuous CSF and that $X_1, \ldots, X_n$ are independent, absolutely continuous random variables.

(i) If for all $\underline{y} \in P_1$, $y_j = 1$ for some $j \neq i$, then $\lim_{\alpha \to 1} R_i(\alpha) = 0$.

(ii) If for all $\underline{w} \in K_0$, $w_j = 0$ for some $j \neq i$, then $\lim_{\alpha \to 0} R_i(\alpha) = 0$.

Theorem

Let γ be a continuous CSF such that $\mu(U_\alpha) > 0$ for all $\alpha \in (0,1)$ and suppose that $X_1, \ldots, X_n$ are independent, absolutely continuous random variables. Then $R_i(\alpha) = 0$ for $\alpha \in (0,1)$ if and only if $y_i = 0$ for every $\underline{y} \in P_\alpha$ for which $\mu(U(\underline{y})) > 0$ where $U(\underline{y}) = \{\underline{x} | \underline{x} \geq \underline{y}\}$ (μ denotes Lebesgue measure).

5. CANNIBALIZATION

In many instances, repair facilities or spare parts may not be immediately available so that system reliability can only be enhanced by extracting needed components from another part of the system. This process is known as cannibalization.

Suppose that the n components of (C,γ) can be divided into N types $(1 \leq N \leq n)$ and that only components of the same type can be interchanged, thereby generating equivalence classes, $Q_1,\ldots,Q_N$ say. Let $S(r)$ denote the symmetric group of order r and let $\underline{x}^A$ denote $\{x_i \in \underline{x} | i \in A \subset C\}$ for $\underline{x} \in \Delta$. We write $\underline{x}R\underline{y}$ if there exists a $P_i \in S(|Q_i|)$ such that $\underline{x}^{Q_i} = P_i\underline{y}^{Q_i}$ for $i=1,2,\ldots,N$ for $\underline{x},\underline{y} \in \Delta$. Let $[\underline{x}]$ denote the equivalence class generated by the natural quotient mapping on Δ/R by a fixed vector $\underline{x}$.

Definition

A cannibalization is a transformation $T: \Delta \to \Delta$ such that $T\underline{x} \in [\underline{x}]$ for all $\underline{x} \in \Delta$.

Let $\mathcal{T}$ denote the set of all cannibalizations of (C,γ) for given equivalence classes $Q_1,\ldots,Q_N$ of component types.

Definition

The cannibalization T is admissible (for $\underline{x}$) if $\gamma(T\underline{x}) \geq \gamma(T'\underline{x})$ for all $T' \in \mathcal{T}$.

The class of all admissible cannibalizations for a fixed vector $\underline{x} \in \Delta$ is denoted $\mathcal{T}^* = \mathcal{T}^*(\underline{x})$.

Definition

Suppose that the null cannibalization is admissible for all $\underline{x} \in \Delta$. Then the CSF γ is said to be cannibalization-invariant. If γ is cannibalization-invariant for all possible partitions $Q_1,\ldots,Q_N$ $(N \geq 2)$ of C, γ is said to be uniformly cannibalization-invariant.

Theorem

A CSF γ is uniformly cannibalization-invariant if and only if $\gamma(\underline{x}) = \gamma(P\underline{x})$ for all $P \in S(n)$.

For any given CSF γ and component types $Q_1,\ldots,Q_N$, the set of admissible cannibalizations induces a new function

$$\gamma^*(\underline{x}) = \max_{\underline{y}\in[\underline{x}]} \gamma(\underline{y}) = \gamma(T\underline{x}) \text{ for } T \in \mathcal{T}^*(\underline{x})$$

which we call the <u>cannibalized</u> <u>structure</u> <u>*function*</u>.

<u>Theorem</u>

γ^* is a CSF.

<u>Theorem</u>

If γ is a continuous CSF, then γ^* is continuous.

However, γ^* does not necessarily inherit the image or the coherency of its progenitor γ^*.

<u>Theorem</u>

If γ is a continuous CSF, then Im γ^* = Im γ.

The following theorem extends the Block-Savits decomposition to the cannibalized CSF γ^*.

<u>Theorem</u> (Decomposition of Cannibalized CSFs)

Let γ be a right-continuous CSF. Then we have the representation

$$\gamma^*(\underline{x}) = \int_0^1 \max_{\underline{z}\in P^*_\alpha} \min_{1\leq i\leq n} I_{\{x_i \geq z_i\}} d\alpha$$

where

$$P^*_\alpha = R[\bigcup_{\underline{x}\in\Delta} \bigcup_{T\in\mathcal{T}^*(\underline{x})} \{\underline{y}|T\underline{y}\in P_\alpha\}]$$

and where

$$R[A] = A - \bigcup_{\underline{y}\in A} \{\underline{x}\in A|\underline{x}>\underline{y}\}.$$

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BLOCK, H.W. and SAVITS, T. H. (1984). "Continuous Multistate Structure Functions", Operat. Res., 32, 703-714.

APPENDIX

Papers in Technical Journals

"Modules of Continuum Structures" (with Chul Kim) in Reliability and Quality Control ed. A. P. Basu, pub. North-Holland, pp. 57-68.

"Bounding the Stochastic Performance of Continuum Structure Functions I" (with Chul Kim). To appear in Journal of Applied Probability, Vol. 23, No. 3, 1986.

"Bounding the Stochastic Performance of Continuum Structure Functions II" (with Chul Kim). To appear in Journal of Applied Probability, Vol. 24, No. 3, 1987.

"Reliability Importance for Continuum Structure Functions" (with Chul Kim). To appear in Journal of Applied Probability, Vol. 24, No. 3, 1987.

"Axiomatic Characterizations of Continuum Structure Functions" (with Chul Kim). Submitted to Mathematika.

"On the Theory of Cannibalization". Submitted to Mathematical Proceedings of the Cambridge Philosophical Society.

"A Note on Information and Censored Absolutely Continuous Random Variables". Submitted to Statistics & Probability Letters.

Associated Personnel

Chul Kim — A former doctoral student of the PI's. Ph.D. awarded in August 1985 for his thesis "Continuum Structure Functions: Modules, Bounds, Axiomatization and Reliability Importance".

Seung Min Lee — A doctoral student of the PI's. His thesis title is "An Approximation Theorem for a Class of Upper Semicontinuous Functions with Compact Support".

Eui Yong Lee — A doctoral student of the PI's. He is currently working on a stochastic components change with time.

Gregory Chlouverakis — A doctoral student of the PI's. His thesis topic is yet to be decided.

Papers Presented at Conferences

"Continuum Structure Functions". Invited presentation at AFOSR Reliability Workshop, Virginia, June 1985.

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